QUANTUM THEORY OF THE CLASSICAL

Wojciech Hubert Zurek
(to be continued by Mike Zwolak and Jess Ridel)

“BEYOND DECOHERENCE”

Quantum Jumps (discrete outcomes from unitary evolutions)
Quantum Origins of Probability (from entanglement)
Quantum Darwinism (the origin of “objective reality”)

WHZ, Nature Physics March 2009
Quantum Theory of the Classical

PROGRAM: Take the core “quantum” postulates of textbook quantum theory and investigate whether they can account for the quantum-classical transition.

0. State of a composite system is a vector in the tensor product of constituent Hilbert spaces. 

1. States correspond to vectors in Hilbert space. (“Quantum Superposition Principle”)

2. Evolutions are unitary (e.g. generated by Schroedinger equation). (“Unitarity”)

3. Immediate repetition of a measurement yields the same outcome. (“Repeatability”)

4. (a) Outcomes are restricted to orthonormal states \( \{|s_k>\} \) (eigenstates of the measured observable). (b) One outcome is seen each time.

5. Probability of an outcome \( |s_k> \) given state \( |f> \) is \( p_k = \langle s_k | f |^2 \). (“Born’s Rule”)

*Postulates according to Dirac
DECOHERENCE, POINTER BASIS, AND EINSELECTION*

\[
\rho_s(t) = \text{Tr}_\mathcal{E} \Phi_{s\mathcal{E}}(t) = \sum_i |\alpha_i|^2 |\sigma_i\rangle\langle\sigma_i|
\]

EINSELECTION* leads to **POINTER STATES**

Stable states, tend to appear near diagonal of \(\rho_s(t)\) after decoherence time; **Pointer states** are effectively classical! They **preserve correlations:** they are **left unperturbed** by the “environmental monitoring”.

*Environment INduced superSELECTION
States in the Hilbert space of an open quantum system evolve from pure into mixed under the influence of both the self-Hamiltonian and the interaction Hamiltonian. They can be sorted according to predictability (e.g. measured by entropy or by purity $h(\rho)$).

$$|\psi\rangle \Rightarrow \rho_\psi(t) \quad \cdots \quad h(\rho_\psi(t)) = Tr\rho_\psi^2(t)$$

$$|\varphi\rangle \Rightarrow \rho_\varphi(t) \quad \cdots \quad h(\rho_\varphi(t)) = Tr\rho_\varphi^2(t)$$

$$|\xi\rangle \Rightarrow \rho_\xi(t) \quad \cdots \quad h(\rho_\xi(t)) = Tr\rho_\xi^2(t)$$
Decoherence restricts stable states (states that can persist, and, therefore, “exist”) to the exceptional…

Pointer states that exist or evolve predictably in spite of the immersion of the system in the environment.

Predictability sieve can be used to ‘sift’ through the Hilbert space of any open quantum system in search of these pointer states.

EINSELECTION (or Environment INDuced superSELECTION) is the process of selection of these preferred pointer states.

For macroscopic systems, decoherence and einselection can be very effective, enforcing ban on Schroedinger cats.

Einselection delineates an effective border that divides quantum from classical, making a point of view similar to Bohr’s Copenhagen Interpretation possible, although starting from a rather different standpoint (i.e., no ab initio classical domain of the universe).

(Zeh, Joos, Paz, Hu, Caldeira, Leggett, Unruh, Gell-Mann, Hartle, Dalvit, Dziarmaga, Schlosshauer, Riedel, Zwolak… …Haroche, Raimond, Brune, Zeilinger, Arndt, Hasselbach…)
EINSELECTION, POINTER BASIS, AND DECOHERENCE

\[ |\Phi_{SE}(0)\rangle = |\psi_S\rangle \otimes |\epsilon_0\rangle = \left( \sum_i \alpha_i |\sigma_i\rangle \right) \otimes |\epsilon_0\rangle \]

Interaction
Entanglement

\[ \sum_i \alpha_i |\sigma_i\rangle \otimes |\epsilon_i\rangle = |\Phi_{SE}(t)\rangle \]

REDUCED DENSITY MATRIX

\[ \rho_{SE}(t) = Tr_\mathcal{E} |\Phi_{SE}(t)\rangle \langle \Phi_{SE}(t)| = \sum_i |\alpha_i|^2 |\sigma_i\rangle \langle \sigma_i| \]

EINSELECTION leads to POINTER STATES

(same states appear on the diagonal of \( \rho_{SE}(t) \) for times long compared to decoherence time)

\[ \left[ H_{SE}, |\sigma_i\rangle \langle \sigma_i| \right] = 0 \]

WE NEED PREFERRED STATES -- "EVENTS"! 

Depends on Born's Rule!!!
Goal

Justify axioms 4&5 using the noncontroversial 0-3.

Understand emergence of “objective classical reality” -- how real states that can be found out by us arise from quantum substrate.

PLAN:

- Why are the measurement outcomes limited to an orthogonal subset of all the possible states in the Hilbert states? (as in “Collapse”)

- Why does “Born’s rule” yield probabilities?

Cannot use decoherence until Born’s rule is established!!!

quantum states that are perturbed by measurements? (“Quantum Darwinism”)

Cannot use decoherence until Born’s rule is established!!!
Quantum Origin of Quantum Jumps

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Theorem: Postulates 0-3 imply breaking of unitary symmetry & single out orthonormal outcomes.

Quantum system \(S\) measured by a quantum apparatus \(A\)

\[ |u> |A_0> \implies |u> |A_u> \]

\[ |v> |A_0> \implies |v> |A_v> \]

By unitarity (#2) scalar product of the total state \((S+A)\) before and after must be the same.

\[ <u|v><A_0|A_0> = <u|v><A_u|A_v> \]

But \(<A_0|A_0> = 1\). So either \(<A_u|A_v> = 1\) (measurement was not successful)

or \(<u|v> = 0 -- the state “jumps” into one of the orthogonal outcomes.

QED!!
Consequences and extensions

- Derivation of “quantum jumps” -- the key to Collapse Postulate from Axioms 0-3: explains why in general one cannot “find out” preexisting states.
- Implies axiom 4a that observables are Hermitean.
- Proof similar to “no cloning” -- information about preexisting states cannot be found out / passed on. (Cloning means making a “perfect copy”. Here the copy need not be perfect.)
- Proof can be extended to when apparatus (or environment) are initially in a mixed state...
- ...as well as to MACROSCOPIC, decohering systems!
- Axiom 3 – repeatability / predictability – is key to the proof!
- Information transfer need not be due to a deliberate measurement: any information transfer that does not perturb outcome states will have to abide by this rule: Pointer states, predictability sieve, and DECOHERENCE.

*Quantum Origin of Quantum Jumps…, PRA Nov ’07; mixed case in PRA May 2013*
Quantum Jumps in Macroscopic Objects

Repeatability in the macroscopic: The state of the system need NOT stay the same. However, it should “look the same”. (For instance, state of the apparatus pointer need not be literally the same, but it should correspond to the same record.) More generally, it should allow the same ACTION: There should be a conditional unitary that as a result of an interaction of the system and apparatuses \( A, A', \ldots \) repeatably yields the same outcomes. Many microstates can lead to same action (can be “same macrostate”, or “same record”). This implies degeneracy:

**“u” subspace**

\[
|u\rangle|A_0\rangle|A_0\rangle\ldots \Rightarrow |\tilde{u}\rangle|A_u\rangle|A_0\rangle\ldots \Rightarrow |\tilde{\tilde{u}}\rangle|A_u\rangle|A_u\rangle\ldots \]

**“v” subspace**

\[
|v\rangle|A_0\rangle|A_0\rangle\ldots \Rightarrow |\tilde{v}\rangle|A_v\rangle|A_0\rangle\ldots \Rightarrow |\tilde{\tilde{v}}\rangle|A_v\rangle|A_v\rangle\ldots \]

Superpositions of states within such degenerate subspaces remain inside these subspaces and results in the same record (repeatability):

\[
(\alpha|v\rangle + \beta|v'\rangle)|A_0\rangle|A_0\rangle\ldots \Rightarrow (\alpha|\tilde{v}\rangle + \beta|\tilde{v}'\rangle)|A_v\rangle|A_0\rangle\ldots \Rightarrow (\alpha|\tilde{\tilde{v}}\rangle + \beta|\tilde{\tilde{v}}'\rangle)|A_v\rangle|A_v\rangle\ldots
\]

Superpositions of states from different subspaces lead to entanglement (NO repeatability)

\[
(\alpha|u\rangle + \beta|v\rangle)|A_0\rangle|A_0\rangle\ldots \Rightarrow (\alpha|\tilde{u}\rangle|A_u\rangle + \beta|\tilde{v}\rangle|A_v\rangle)|A_0\rangle\ldots \Rightarrow \alpha|\tilde{\tilde{u}}\rangle|A_u\rangle|A_u\rangle + \beta|\tilde{\tilde{v}}\rangle|A_v\rangle|A_v\rangle\ldots
\]

REPEATABILITY of RECORDS w/o preservation of the original microstates possible only when the subspaces corresponding to distinct records are orthogonal

MICROSTATE is NOT the same, but the MACROSTATE is!
Orthogonality of mixtures from actionable information

“u” subspace

$\rho^u_S |A_0\rangle \langle A_0| A'_0\rangle \langle A'_0| \ldots \rightarrow \tilde{\rho}^u_S |A_u\rangle \langle A_u| A'_u\rangle \langle A'_u| \ldots,$

$\tilde{\rho}^u_S |A_u\rangle \langle A_u| A'_u\rangle \langle A'_u| \ldots \rightarrow \tilde{\tilde{\rho}}^u_S |A_u\rangle \langle A_u| A'_u\rangle \langle A'_u| \ldots.$

“v” subspace

$\rho^v_S |A_0\rangle \langle A_0| A'_0\rangle \langle A'_0| \ldots \rightarrow \tilde{\rho}^v_S |A_v\rangle \langle A_v| A'_v\rangle \langle A'_v| \ldots,$

$\tilde{\rho}^v_S |A_v\rangle \langle A_v| A'_v\rangle \langle A'_v| \ldots \rightarrow \tilde{\tilde{\rho}}^v_S |A_v\rangle \langle A_v| A'_v\rangle \langle A'_v| \ldots.$

\[
\text{Tr} \rho^u_S \rho^v_S = \text{Tr} \tilde{\rho}^u_S \tilde{\rho}^v_S \langle A_u \mid A_v \rangle^2, \\
\text{Tr} \rho^u_S \rho^v_S = \text{Tr} \tilde{\tilde{\rho}}^u_S \tilde{\tilde{\rho}}^v_S \langle A_u \mid A_v \rangle^4.
\]

The equality of Schmidt-Hilbert norms before and after tagging implies that distinguishable tags ($|\langle A_u \mid A_v \rangle|^2 < 1$) can be imprinted on $A$ only when

\[
\text{Tr} \rho^u_S \rho^v_S = \text{Tr} \tilde{\rho}^u_S \tilde{\rho}^v_S = \text{Tr} \tilde{\tilde{\rho}}^u_S \tilde{\tilde{\rho}}^v_S = \cdots = 0.
\]

Otherwise, $|\langle A_u \mid A_v \rangle|^2 = 1$—no information about $S$ is acquired by $A$, and the measurement is a failure.

REPEATABLY ACCESSIBLY STATES REQUIRE ORTHOGONAL RECORDS
Wave-packet collapse, decoherence, amplification, and actionable information


• Repeatability possible only when the “original” states are distinguishable (orthogonal).

• Actionability of the information is a suitable definition of repeatability when microstate of the original can change but macrostate should not.

• Can use this approach to discuss relation of repeatability to POVM’s (only their records need to be repeatable / distinguishable).

• Actionable mixtures do not mix.

• Can be extended to decohering systems, etc.

• Pure state proof similar to “no cloning” (but perfect copies are not required).

• However, “no broadcasting” is not useful in the “macroscopic” repeatability (microstates can change; only the macrostate should have the same “actionable” consequences).

• Can do everything using pure states only (purification and “Church of Larger Hilbert Space”)

Plan

Derive controversial axioms 4&5 from the noncontroversial 0-3. Understand emergence of “objective classical reality” -- how real states that can be found out by us arise from quantum substrate.

- Why the measurement outcomes are limited to an orthogonal subset of all the possible states in the Hilbert states? **WE HAVE “EVENTS”!**
- Why does “Born’s rule” \( P_k = |\psi_k|^2 \) yield probabilities?
- How can “objective classical reality” -- states we can find out -- arise from the fragile quantum states that are perturbed by measurements? (“Quantum Darwinism”)
I.2 ON THE QUANTUM MECHANICS OF COLLISIONS

[Preliminary communication]†

MAX BORN

Through the investigation of collisions it is argued that quantum mechanics in the Schrödinger form allows one to describe not only stationary states but also quantum jumps.

† This report was originally intended for die Naturwissenschaften, but could not be accepted there for lack of space. I hope that its publication in this journal [Zeitschrift für Physik] does not seem out of place [M.B.].


The calculation gives this result: The scattered wave created by this perturbation has asymptotically at infinity the form:

\[ \psi^{\text{sc}}(x, y, z; q_k) = \sum_{n} \int_{-\infty}^{\infty} d\alpha \int_{-\alpha}^{\infty} d\beta \int_{-\beta}^{\infty} d\gamma \sin \left( \lambda_{n,m}(2x + \beta y + \gamma z + \delta) \right) \psi_m^{\text{e}}(q_k). \]

This means that the perturbation, analyzed at infinity, can be regarded as a superposition of solutions of the unperturbed problem. If one calculates the energy belonging to the wavelength \( \lambda_{n,m} \) according to the de Broglie formula, one finds

where the \( \nu_{n,m} \) are the frequencies.

If one translates this result into terms of particles, only one interpretation is possible. \( \Phi_{n,m}(\alpha, \beta, \gamma) \) gives the probability* for the electron, arriving from the \( z \)-direction, to be thrown out into the direction designated by the angles \( \alpha, \beta, \gamma \), with the phase change \( \delta \). Here its energy \( \tau \) has increased by one quantum \( h\nu_{n,m}^0 \) at the cost of the energy of the atom (collision of the first kind for \( W_n^0 < W_m^0, h\nu_{n,m}^0 < 0 \); collision of the second kind \( W_n^0 > W_m^0, h\nu_{n,m}^0 > 0 \)).

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* Addition in proof: More careful consideration shows that the probability is proportional to the square of the quantity \( \Phi_{n,m} \).
The calculation gives this result: The scattered wave created by ion has asymptotically at infinity the form:

$$\psi_n(x, y, z; q_0) = \sum_{\mu} \int \int_{x, y > 0} d\alpha \Phi_{n, \alpha}(x, \beta, \gamma) \sin \phi_{\alpha n}(2x + \beta y +$$

This means that the perturbation, analyzed at infinity, can be regarded as the position of solutions of the unperturbed problem. If one calculates belonging to the wavelength $\lambda_{n, \alpha}$ according to the de Broglie form

$$W_{n, \alpha} = h\phi_{n, \alpha} + \tau,$$

where the $\phi_{n, \alpha}$ are the frequencies of the unperturbed atom.

If one translates this result into terms of particles, only one interpretation is possible. $\Phi_{n, \alpha}(x, \beta, \gamma)$ gives the probability* for the electron, arriving in the direction, to be thrown out into the direction designated by the angle $\phi_{n, \alpha}$ with the phase change $\delta$. Here its energy $\tau$ has increased by one quantum $h\phi_{n, \alpha}$ at the cost of the energy of the atom (collision of the first kind for $W_n^0 < W_0^0$, $h\phi_{n, \alpha}^0 < 0$; collision of the second kind $W_n^0 > W_0^0$, $h\phi_{n, \alpha}^0 > 0$).

Schrödinger’s quantum mechanics therefore gives quite a definite answer to the question of the effect of the collision; but there is no question of any causal description. One gets no answer to the question, “what is the state after the collision,” but only to the question, “how probable is a specified outcome of the collision” (where naturally the quantum mechanical energy relation must be fulfilled).

Here the whole problem of determinism comes up. From the standpoint of our quantum mechanics there is no quantity which in any individual case causally fixes the consequence of the collision; but also experimentally we have so far no reason to believe that there are some inner properties of the atom which condition a definite outcome for the collision. Ought we to hope later to discover such properties (like phases or the internal atomic motions) and determine them in individual cases? Or ought we to believe that the agreement of theory and experiment—as to the impossibility of prescribing conditions for a causal evolution—is a pre-established harmony founded on the nonexistence of such conditions? I myself am inclined to give up determinism in the world of atoms. But that is a philosophical question for which physical arguments alone are not decisive.

* Addition in proof: More careful consideration shows that the probability is proportional to the square of the quantity $\Phi_{n, \alpha}$. 

* Addition in proof: More careful consideration shows that the probability is proportional to the square of the quantity $\Phi_{n, \alpha}$.
ENVARIANCE

(Entanglement-Assisted Invariance)

DEFINITION:

Consider a composite quantum object consisting of system $S$ and environment $E$. When the combined state $\psi_{SE}$ is transformed by:

$$U_S = u_s \otimes 1_E$$

but can be “untransformed” by acting solely on $E$, that is, if there exists:

$$U_E = 1_S \otimes u_E$$

then $\psi_{SE}$ is ENVARIANT with respect to $u_s$.

$$U_E (U_S |\psi_{SE}\rangle) = U_E |\varphi_{SE}\rangle = |\psi_{SE}\rangle$$

Envariance is a property of $U_S$ and the joint state $\psi_{SE}$ of two systems, $S$ & $E$. 
ENTANGLED STATE AS AN EXAMPLE OF ENVARIENCE:

Schmidt decomposition:

$$\left| \psi_{SE} \right\rangle = \sum_{k=1}^{N} \alpha_k \left| s_k \right\rangle \left| \varepsilon_k \right\rangle$$

Above Schmidt states $\left| s_k \right\rangle$, $\left| \varepsilon_k \right\rangle$ are orthonormal and $\alpha_k$ complex.

Lemma 1: Unitary transformations with Schmidt eigenstates:

$$u_S(s_k) = \sum_{k=1}^{N} \exp(i\phi_k) \left| s_k \right\rangle \left\langle s_k \right|$$

leave $\psi_{SE}$ invariant.

Proof: $u_S(s_k)\left| \psi_{SE} \right\rangle = \sum_{k=1}^{N} \alpha_k \exp(i\phi_k) \left| s_k \right\rangle \left\langle \varepsilon_k \right|$, $u_\varepsilon(\varepsilon_k) = \sum_{k=1}^{N} \exp\{i(-\phi_k + 2\pi l_k)\} \left| s_k \right\rangle \left\langle \varepsilon_k \right|$

$$u_\varepsilon(\varepsilon_k)\{u_S(s_k)\left| \psi_{SE} \right\rangle\} = \sum_{k=1}^{N} \alpha_k \exp\{i(\phi_k - \phi_k + 2\pi l_k)\} \left| s_k \right\rangle \left\langle \varepsilon_k \right| = \sum_{k=1}^{N} \alpha_k \left| s_k \right\rangle \left\langle \varepsilon_k \right| = \left| \psi_{SE} \right\rangle$$

LOCALLY, SCHMIDT PHASES DO NOT MATTER: DECOHERENCE!!!
PHASE ENVARIANCE THEOREM

Quantum Fact 1: Unitary transformations must act on the system to alter its state (if they act only somewhere else, system is not effected).

Quantum Fact 2: The state of the system is all that is necessary/available to predict measurement outcomes (including their probabilities).

Quantum Fact 3: A state of the composite system is all that is needed/available to determine the state of the system.

Moreover, “entanglement happens”:

\[ |\psi_{SE}\rangle \propto \sum_{k=1}^{N} \alpha_k |s_k\rangle |e_k\rangle \]

THEOREM 1: State (and probabilities) of S alone can depend only on the absolute values of Schmidt coefficients, and not on their phases.

Proof: Phases of \( \alpha_k \) can be changed by acting on S alone. But the state of the whole can be restored by acting only on E (by envariance). So, by Fact 3, state of S can be also restored. However (by Fact 1) state of S cannot be changed by acting only on E. So (by Fact 2) change of phases of Schmidt coefficients could not have affected the state of S! QED.

- By phase envariance \( \{|\alpha_k\rangle, |s_k\rangle\} \) must provide a complete local description of the state of the system S alone. Same info as reduced density matrix!!!
Envariance of entangled states: the case of equal coefficients

$$|\psi_{s\varepsilon}\rangle \propto \sum_{k=1}^{N} \exp(i\phi_k)|s_k\rangle|\varepsilon_k\rangle$$

In this case ANY orthonormal basis is Schmidt. In particular, in the Hilbert subspace spanned by any two \(\{|s_k\rangle, |s_l\rangle\}\) one can define a Hadamard basis;

$$|\pm\rangle = (|s_k\rangle \pm |s_l\rangle)/\sqrt{2}$$

This can be used to generate ‘new kind’ of envariant transformations:

\[c)\]

\[\begin{align*}
|\spadesuit\rangle \pm |\diamondsuit\rangle &= |\heartsuit\rangle \pm |\clubsuit\rangle \\
|\heartsuit\rangle \pm |\spadesuit\rangle &= |\diamondsuit\rangle \pm |\clubsuit\rangle \\
|\diamondsuit\rangle \pm |\heartsuit\rangle &= |\spadesuit\rangle \pm |\clubsuit\rangle \\
|\clubsuit\rangle \pm |\diamondsuit\rangle &= |\spadesuit\rangle \pm |\heartsuit\rangle
\end{align*}\]
Probability of envariantly swappable states

\[ |\psi_{\mathcal{E}}\rangle \propto \sum_{k=1}^{N} \exp(i\phi_k) |s_k\rangle |\varepsilon_k\rangle \]

By the Phase Envariance Theorem the set of pairs \(|\alpha_k\rangle, |s_k\rangle\) provides a complete description of \(\mathcal{S}\). But all \(|\alpha_k\rangle\) are equal.

With additional assumption about probabilities (e.g., perfect correlation as on the previous transparency), one can prove

**THEOREM 2:** Probabilities of envariantly swappable states are equal.

(a) “Pedantic assumption”; when states get swapped, so do probabilities;
(b) When the state of the system does not change under any unitary in a part of its Hilbert space, probabilities of any set of basis states are equal.
(c) Because there is one-to-one correlation between \(|s_k\rangle, |\varepsilon_k\rangle\)

Therefore, by normalization:

\[ P_k = \frac{1}{N} \quad \forall k \]
Special case with **unequal** coefficients

Consider system $\mathcal{S}$ with two states $\{\ket{0}, \ket{2}\}$

The environment $\mathcal{E}$ has three states $\{\ket{0}, \ket{1}, \ket{2}\}$ and $\ket{+} = (\ket{0} + \ket{1})/\sqrt{2}$

$$\ket{\psi_{\mathcal{S}\mathcal{E}}} = \sqrt{\frac{2}{3}} \ket{0} \ket{+} + \sqrt{\frac{1}{3}} \ket{2} \ket{2}$$

An auxiliary environment $\mathcal{E}'$ interacts with $\mathcal{E}$ so that:

$$\ket{\psi_{\mathcal{S}\mathcal{E}}} \ket{\mathcal{E}'_0} = \left(\sqrt{\frac{2}{3}} \ket{0} \ket{+} + \sqrt{\frac{1}{3}} \ket{2} \ket{2}\right) \ket{0} \Rightarrow \sqrt{\frac{2}{3}} \ket{0} \left(\ket{0} \ket{0} + \ket{1} \ket{1}\right)/\sqrt{2} + \sqrt{\frac{1}{3}} \ket{2} \ket{2} \ket{2} =$$

$$= \left(\ket{0} \ket{0} \ket{0} + \ket{0} \ket{1} \ket{1} + \ket{2} \ket{2} \ket{2}\right)/\sqrt{3}$$

States $\ket{0} \ket{0}$, $\ket{0} \ket{1}$, $\ket{2} \ket{2}$ have equal coefficients. Therefore, each of them has probability of $1/3$. Consequently:

$$p(0) = p(0,0) + p(0,1) = \frac{2}{3}, \quad \text{and} \quad p(2) = \frac{1}{3}.$$  

**..... BORN’s RULE!!!**

*no need to assume additivity!* (p(0)=1-p(2)!)
Probabilities from Envariance

The case of commensurate probabilities: 
\[ |\psi_{SE}\rangle = \sum_{k=1}^{N} \sqrt{\frac{m_k}{M}} |s_k\rangle |\varepsilon_k\rangle \]

Attach the auxiliary “counter” environment \( \mathcal{C} \):

\[ |\psi_{SE}\rangle |e'_0\rangle = \left( \sum_{k=1}^{N} \sqrt{\frac{m_k}{M}} |s_k\rangle \right) \left( \sum_{j_k=1}^{m_k} \frac{1}{\sqrt{m_k}} |e_{j_k}\rangle \right) |c_0\rangle \Rightarrow \]

\[ \Rightarrow \frac{1}{\sqrt{M}} \sum_{j=1}^{M} |s_k(j)\rangle |e_{j}\rangle |c_{j}\rangle \]

**THEOREM 3:** The case with commensurate probabilities can be reduced to the case with equal probabilities. **BORN’s RULE follows:**

\[ p_j = \frac{1}{M}, \quad p_k = \sum_{j_k=1}^{m_k} p_{j_k} = \frac{m_k}{M} = |\alpha_k|^2 \]

General case -- by continuity. QED.
ENVARIANCE* -- SUMMARY

1. New symmetry - ENVARIANCE - of joint states of quantum systems. It is related to causality.

2. In quantum physics perfect knowledge of the whole may imply complete ignorance of a part.

3. BORN’s RULE follows as a consequence of envariance.

4. Relative frequency interpretation of probabilities naturally follows.

5. Envariance supplies a new foundation for environment - induced superselection, decoherence, quantum statistical physics, etc., by justifying the form and interpretation of reduced density matrices.

*WHZ, PRL 90, 120404; RMP 75, 715 (2003); PRA 71, 052105 (2005); PRL 106, 250402 (2011).
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Derive controversial axioms 4&5 from the noncontroversial 0-3. Understand emergence of “objective classical reality” -- how real states that can be found out by us arise from quantum substrate.

- Why the measurement outcomes are limited to an orthogonal subset of all the possible states in the Hilbert states?
- Why does “Born’s rule” yield probabilities?
- How can “objective classical reality” -- states we can find out -- arise from the fragile quantum states that are perturbed by measurements? (“Quantum Darwinism”)